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Determination of All Primitive Collineation Groups in More than Four Variables which Contain Homologies.

BY HOWARD H. MITCHELL.

In a recent paper (*Proc. London Math. Soc.*, Series 2, Vol. X, Part 4, Nov., 1911) Burnside determined all the finite collineation groups in n variables with rational coefficients which contain the symmetric group on those variables, *i. e.*, the group generated by all transformations of the form

$$x'_i = x_{i+1}, \ x'_{i+1} = x_i, \ x'_j = x_j \ (i = 1, 2, \ldots, n-1; \ j \neq i, \ i+1).$$

The groups of this sort (for $n > 4$) which are primitive (in the sense of Blichfeldt) he found to be groups of order $n+1!$ (for any n), and groups of order $2^7 \cdot 3^4 \cdot 5$, $2^9 \cdot 3^4 \cdot 5 \cdot 7$, and $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ for $n = 6, 7, 8$ respectively.

The group of order $2^7 . 3^4 \cdot 5$ (for $n = 6$) is isomorphic with the group of the equation for the 27 lines on a cubic surface and was first exhibited in explicit form by Burkhardt (*Math. Ann.*, Bd. XLI, pp. 320–326). The groups for $n = 7, 8$ had not been noticed before. That of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ is shown by Burnside to give a representation of the group of the 28 bitangents to a quartic curve. He states that the one of order $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ is isomorphic with Jordan's hypo-abelian group on eight variables.

The author solves a more general problem, which is the determination of of all primitive groups in n (> 4) variables which contain homologies, *i. e.*, transformations all of whose multipliers are equal with the exception of one. A transformation in the symmetric group on the variables which interchanges two of them and leaves the rest unaltered has -1 for one multiplier and $+1$ for each of the others, being therefore an homology of period 2. The determination will be made in such a way as to apply equally well to the groups in a general modular space,[*] whose orders are not divisible by the modulus.

The results for $n \leq 4$ are already known. For $n = 3$[†] all primitive groups contain homologies, that of order 216 containing homologies of period 3 and of period 2, and those of order 72, 36, 168, 60, 360 homologies of period 2. For $n = 4$* the primitive groups of this sort are of order 25920, 120, 288, 576, 1152, 1920, 11520, 7200. That of order 25920 contains homologies of period 3 and the others homologies of period 2.

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[*] Veblen and Bussey, "Finite Projective Geometries," *Trans. Am. Math. Soc.*, Vol. VII (1906), pp. 241–259.

[†] See papers by Blichfeldt, *Trans. Am. Math. Soc.*, Vol. V (1904), pp. 310–325; *Math. Ann.*, Bd. LXIII (1907), pp. 552–572; also one by the author, *Trans. Am. Math. Soc.*, Vol. XII (1911), pp. 207–242.

For $n > 4$ there are no primitive groups containing homologies of period greater than 2. In addition to the primitive groups in Burnside's list there are two others which contain homologies of period 2. These are of order $2^6 \cdot 3^4 \cdot 5$ and $2^8 \cdot 3^6 \cdot 5 \cdot 7$ for $n = 5, 6$ respectively. The first of these has been discussed by Burkhardt (*Math. Ann.*, Bd. XXXVIII, pp. 185–224), whereas the latter does not seem to have been noticed. The former is isomorphic with the well-known simple group of that order, and the latter is (1, 2) isomorphic with the first orthogonal group on six indices with modulus 3.†

§ 1. THEOREM 1. *No primitive group in more than four variables contains homologies of higher period than* 2.

Two homologies which are not commutative and a third not commutative with the group generated by the first two and with center not on the line of centers must generate a group which permutes the points of the plane containing the centers. There is no possible group in the plane if the homologies are of higher period than 3. If they are of period 3, the group generated must be of order $3 \cdot 216$.

An homology not commutative with this group of order $3 \cdot 216$ must generate with it a group of order $6 \cdot 25920$. The invariant C_3 is in the group generated by any four mutually commutative homologies. The invariant C_2 may be obtained as the product of two commutative C_2, such as $x'_i = -x_i$, $x'_j = x_j$ $(i = 1, 2;\ j > 2)$, and $x'_i = -x_i$, $x'_j = x_j$ $(i = 3, 4;\ j \neq 3, 4)$.

All other homologies of period 3 must then be commutative with this group. For otherwise there would be on the points of a line a C_6 and a C_3 not in the C_6. But no group on the line can contain such cyclic groups, Hence no primitive group in more than four variables can contain homologies of period 3.

THEOREM 2. *No primitive group in more than four variables contains two homologies of period* 2 *whose product is of period greater than* 3.

Two homologies of period 2 which are not commutative and a third which is not commutative with the group generated by the first two and whose center is not on the line of centers must generate a group permuting the points of the plane containing the centers. The group in this plane may be primitive or it may permute in all six ways the vertices of a triangle. In the latter case only one triangle is left invariant in this plane provided there are homologies present whose product is of period greater than 3.

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* See Bagnera, *Rendiconti del Circolo Matematico di Palermo,* T. XIX (1905), pp 1–56; Blichfeldt, *Math. Ann.*, Bd. LX (1905), pp. 204–231.

† Jordan, *Traité des Substitutions*, pp. 161–170; Dickson, *Linear Groups*, Chap. VII.

Consider now an homology not commutative with this group and with center not in the plane of centers. The group then generated in the S_3 which contains the centers may be either primitive or imprimitive. In the latter case it must permute the vertices of a tetrahedron, three of the vertices of which are vertices of the invariant triangle in the plane. For the latter group carries every point which does not remain invariant under it and which does not lie in the plane of centers into more than four positions, provided there are homologies present whose product is of period greater than 3. Hence any homology with center in the S_3 which is not commutative with the group and whose center does not lie in the plane of centers must leave fixed two of the vertices of the triangle and carry the third into a point which remains invariant under the group.

In case there is no primitive group in an S_3 whatever, every homology must be of this sort. Hence an homology with center not in the S_3 and not commutative with the group permuting the points of the S_3 must leave fixed at least two of the vertices of each of the four triangles of the tetrahedron and hence must leave fixed three of the four. It must carry the fourth into a point which remains invariant under each of the four subgroups and hence into a point which remains invariant under the whole group permuting the points of the S_3. Hence the group generated by homologies with centers in an S_4 must permute five points in that S_4.

We may then prove readily by induction that the only possible group generated by homologies with centers in an S_{n-1} must permute n points in that S_{n-1}.

It follows therefore that there must be a primitive group generated by homologies with centers in an S_3. Those which contain homologies whose product is of period greater than 3 are of order 576, 1152, 1920, 11520, 7200.* In every case there is an invariant C_2 obtained as the product of four mutually commutative homologies. The first three of these groups contain homologies each commutative with a group of order $2 \cdot 24$, that of order 11520 contains homologies each commutative with a group of order $2 \cdot 96$, and the one of order 7200 contains homologies each commutative with a group of order $2 \cdot 60$.

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* *Loc. cit.*

In any larger space containing the S_3 all these groups are of double the order. We denote the whole group by G and by H a subgroup of order $2 \cdot 24$, $2 \cdot 96$, or $2 \cdot 60$ generated by the homologies with centers in a plane.

An homology not commutative with G and with center not in the S_3 must generate with G a group permuting the points of an S_4. In this S_4 the C_2 which is invariant under G is an homology, which we denote by h_g. The homology which is invariant under H but not under G we denote by h_h.

In the S_4 there must certainly be centers which lie in neither of the two S_3 determined by the plane of centers of H and the center of h_g or h_h. An homology with such a center must generate with H either a primitive group in the S_3 containing the centers or else a group permuting the vertices of a tetrahedron. In either case it must contain an homology commutative with H with center at the point where this S_3 meets the line joining the centers of h_g and h_h.

But there can be no other centers of homologies on this line, for it is contained by planes in which the group of the points is a dihedral G_{12}, whereas no primitive group of the points in a plane contains a dihedral group of higher order than 10. Such a G_{12} is generated by h_g, h_h, and an homology which is commutative with h_g and whose product by h_h is of period 3. Hence no primitive group in more than four variables contains homologies whose product is of period greater than 3.

§ 2. *Groups in which every homology is commutative with at least one homology in each dihedral G_6.*

A dihedral G_6 generated by two homologies and an homology not commutative with this G_6 must generate a group permuting the points of the plane containing the centers. The only possible groups in a plane in which the product of no two homologies is of higher period than 3 are of order 24 and 18. The former permutes the vertices of one triangle in that plane and the latter permutes the vertices of each of four triangles. If the group in the plane is of order 18, the group in any larger space containing this plane is of order $3 \cdot 18$. We first consider groups which do not contain such groups as subgroups. Hence we may suppose that every homology is commutative with at least one in each dihedral G_6, since this is the case in a G_{24}.

THEOREM 3. *The only primitive collineation groups in n (>4) variables, which contain homologies and in which every homology is commutative with at least one in each dihedral G_6, are groups of order $n+1!$ for any n, and groups of order $2^7 \cdot 3^4 \cdot 5$, $2^9 \cdot 3^4 \cdot 5 \cdot 7$ and $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ for $n = 6, 7, 8$ respectively.*

Consider a G_{24}, which we represent as the symmetric group in the first four variables. In a primitive group there must be an homology not commutative with this group. Since it must be commutative with at least one homology in each dihedral G_6, its center must be conjugate under the G_{24} with a point such as $(a\,a\,a\,b\,c\,\ldots.)$ or $(a\,a\,b\,b\,c\,\ldots.)$, where $a \neq b$.

In case the center of every homology which is not commutative with the G_{24} is of the first type, the only primitive groups in n variables which exist are of order $n+1!$ This may be proved by induction. Consider the symmetric group in n variables. The center of an homology not commutative with it must be conjugate under the symmetric group with a point such as

$$(\xi_1 + \xi_2 + \ldots. + \xi_{n-1})\, a + \xi_n b + \ldots. = 0,^*$$

since three of any four of the first n coordinates must be equal.

We first suppose that it is linearly dependent on the centers in the symmetric group. It is then

$$(\xi_1 + \xi_2 + \ldots. + \xi_{n-1}) + (1-n)\,\xi_n = 0.$$

We may suppose that the invariant linear system of this homology is

$$(x_1 + x_2 + \ldots. + x_{n-1}) + (1-n)\,x_n = 0,$$

since we are concerned only with the group in the S_{n-2} which contains the centers. But the product of this homology by

$$x'_{n-1} = x_n,\ x'_n = x_{n-1},\ x'_j = x_j\ \ (j \neq n-1,\, n)$$

is not of period 3 since the centers and invariant linear systems do not cut out the proper cross-ratio on the line joining the centers for any positive integral value of n.†

We now suppose that the center is linearly independent of the centers in the symmetric group. We may then by a transformation commutative with the symmetric group transform it into $\xi_n - \xi_{n+1} = 0$. We find that the invariant linear system of points may be chosen as $x_n - x_{n+1} = 0$. The homology is then

$$x'_n = x_{n+1},\ x'_{n+1} = x_n,\ x'_j = x_j\ \ (j \neq n,\, n+1).$$

We then have the symmetric group on $n+1$ variables. It is of course primitive only if represented on n variables, which is possible by reason of the invariant linear relation $x_1 + x_2 + \ldots. + x_{n+1} = 0$.

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* We shall find it convenient to use a dual coordinate system.

† It is perhaps of interest to note that if the coefficients be reduced by any odd prime as a modulus that is contained in $n+1$, the product of these transformations is of period 3. In such a case a group of order $n+1!$ exists for which only $n-1$ independent variables are necessary. For example, we may exhibit the $G_{5!}$ in three variables by using the modulus 5, and the $G_{6!}$ in four variables by using the modulus 3 (cf. Dickson, *Trans. Am. Math. Soc.*, Vol. IX, pp. 121–148).

We may suppose therefore that any other primitive group in which every homology is commutative with at least one in each dihedral G_6 contains the symmetric group on the first four variables and an homology with center $(a\,a\,b\,b\,c\,\ldots.)$. We may take this center to be $\xi_1+\xi_2=0$. The invariant linear system of the homology we may take to be $x_1+x_2=0$. The homology is then $x_1'=-x_2$, $x_2'=-x_1$, $x_j'=x_j$ $(j>2)$. The group generated is of order $2^3\cdot 4!$ in the $n(>4)$ variables. It permutes the vertices of each of three tetrahedra in $x_5=x_6=\ldots.=0$, one vertex of each being

$$\xi_1=0,\ \xi_1+\xi_2+\xi_3+\xi_4=0,\ -\xi_1+\xi_2+\xi_3+\xi_4=0.$$

Consider an homology not commutative with this group. In order that it be commutative with at least one in each dihedral G_6 we find that of the first four coordinates of its center either three must be 0 or else the ratios of the four must be ± 1. There can be no center of either sort linearly dependent on the centers in the above group without involving homologies whose product is of period 4. Hence we suppose the existence of a center linearly independent of them. Since the group of order $2\cdot 96$ is invariant under groups of order $2\cdot 288$ and $2\cdot 576$ permuting the three tetrahedra mentioned above, we may take a center to be of the first type, *e. g.*, $\xi_4-\xi_5=0$. We find that the invariant linear system may be taken to be $x_4-x_5=0$. The homology is then $x_4'=x_5$, $x_5'=x_4$, $x_j'=x_j$ $(j\neq 4,5)$. A group is generated of order $2^4\cdot 5!$.

Consider an homology not commutative with this group. Of the first five coordinates of its center either four must be 0 or else the ratios of the five must be ± 1. Since the group is invariant under a transformation which changes the sign of an arbitrary number of variables and leaves the rest unaltered, we may in the latter case choose the first five coordinates all as $+1$.

We find readily that there can be no more centers linearly dependent on the centers in the above group. We therefore suppose the existence of an homology with center linearly independent of them. If all the centers of this sort are such that of the first five coordinates four are 0, it may readily be shown that no primitive group in n variables is possible. We therefore take an homology with center $\xi_1+\xi_2+\ldots.+\xi_8=0$. We find then that its invariant system of points may be taken to be $x_1+x_2+\ldots.+x_8=0$. This homology may be written:

$$x_1'+x_2'+\ldots.+x_8'=-(x_1+x_2+\ldots.+x_8),$$
$$x_{i+1}'-x_i'=x_{i+1}-x_i,\ x_j'=x_j\ (i<8,\ j>8).$$

A group is then generated, permuting 36 centers of homologies. The group commutative with one of the homologies is of order $2\cdot 6!$ and the whole group of order $2\cdot 6!\cdot 36 = 2^7\cdot 3^4\cdot 5$. It is the well-known group which gives a representation of the group of the 27 lines on a cubic surface. It requires of course only six independent variables for its representation, the above choice of coordinates having been made for convenience in exhibiting the larger groups.

Under this group the subgroup commutative with $T: x_i' = -x_i,\ x_j' = x_j$ $(i \leqq 4,\ j > 4)$ is of order $2\cdot 576$. This subgroup permutes the three invariant tetrahedra of the group of order $2\cdot 96$. On the points in $x_1 = x_2 = x_3 = x_4 = 0$ it is of order 6. Each of the six transformations on these points corresponds to a particular permutation of the three tetrahedra. To one which leaves fixed the tetrahedron $\xi_1 = 0,\ \xi_2 = 0,\ \xi_3 = 0,\ \xi_4 = 0$ and interchanges the other two, there corresponds the transformation $U: x_5' = -x_5,\ x_j' = x_j\ (j > 5)$, and to one which leaves fixed the tetrahedron $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0,\ \xi_1 + \xi_2 - \xi_3 - \xi_4 = 0,$ $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0,\ \xi_1 - \xi_2 - \xi_3 + \xi_4 = 0$ and interchanges the other two, there corresponds the transformation $V: x_5' + x_6' + x_7' + x_8' = -(x_5 + x_6 + x_7 + x_8),$ $x_{i+1}' - x_i' = x_{i+1} - x_i,\ x_j' = x_j\ (i = 5, 6, 7;\ j > 8)$. Of the 36 centers of homologies 12 are in $x_5 = x_6 = \ldots. = 0$ and the other 24 lie by pairs on the 12 lines joining the centers of U, V, UVU in $x_1 = x_2 = x_3 = x_4 = 0$ to the vertices of the corresponding tetrahedra in $x_5 = x_6 = \ldots. = 0$.

Consider now an homology not commutative with the whole group. Of the first five coordinates of its center either four must be 0 or else the ratios of the five must be ± 1. Under the group of order $2\cdot 576$ commutative with T a center of the latter type will be conjugate with a center of the former type. Since the symmetric group on the first five variables is present, it will be conjugate with a center in $x_1 = x_2 = x_3 = x_4 = 0$. An homology of this sort must be commutative with all those with centers in $x_5 = x_6 = \ldots. = 0$. Hence under the group generated by any such homologies no two of the three C_2, U, V, UVU can be conjugate. For in that case there would be centers on lines joining $\xi_5 = 0$ to points in $x_5 = x_6 = \ldots. = 0$ other than the four vertices of the corresponding tetrahedron. But there can be no more centers in the S_4 determined by $\xi_5 = 0$ and that tetrahedron.

There can be no homology with center on the line containing the centers of U, V, UVU in $x_1 = x_2 = x_3 = x_4 = 0$. For the only such homologies which could generate a group under which these three centers were not conjugate would be two with centers at the two points interchanged by the dihedral G_6. But neither of these points lies in

$$x_4 + x_5 = 0,\ x_1 + x_2 + \ldots. + x_8 = 0,\ x_1 + x_2 + x_3 - x_4 - x_5 + x_6 + x_7 + x_8 = 0,$$

and hence neither homology could be commutative with any one of the three in the dihedral G_6 in which these three are the invariant linear systems.

Hence we may suppose the existence of centers of homologies in $x_1 = x_2 = x_3 = x_4 = 0$, but not on the line containing the centers of U, V, UVU. The group generated by one such homology together with U, V, UVU must have a self-conjugate subgroup containing the homologies under which U, V, UVU are not conjugate. Hence the whole group must be of order $2 \cdot 24$ and the homologies must generate its invariant subgroup of order $2 \cdot 4$. One of the three homologies must be commutative with U and we may take its center to be $\xi_7 + \xi_8 = 0$. The centers of the other two are then $\xi_5 + \xi_6 = 0$ and $\xi_5 - \xi_6 = 0$. The invariant linear system of the homology with center $\xi_7 + \xi_8 = 0$ may then be chosen as $x_7 + x_8 = 0$. It is therefore $x'_7 = -x_8$, $x'_8 = -x_7$, $x'_j = x_j$ $(j \neq 7, 8)$.

A group is then generated permuting 63 centers of homologies. Of these 63 twelve are in $x_5 = x_6 = \ldots . = 0$ and three in $x_1 = x_2 = x_3 = x_4 = 0$, and the other 48 lie by pairs on 24 lines joining the vertices of the three tetrahedra in $x_5 = x_6 = \ldots . = 0$ with three corresponding pairs of points in $x_1 = x_2 = x_3 = x_4 = 0$. The group commutative with one of the 63 homologies is of order $2^6 \cdot 6!$. It contains, however, the C_2, $x'_i = -x_i$, $x'_7 = -x_8$, $x'_8 = -x_7$, $x'_j = x_j$ $(i < 7,\ j > 8)$, which is invariant under the whole group. The group of collineations in the S_6 which contains the centers of homologies is then of order $2^5 \cdot 6! \cdot 63 = 2^9 \cdot 3^4 \cdot 5 \cdot 7$.

This is one of the two groups found by Burnside (*loc. cit.*), and it is shown by him to give a representation of the group of the 28 bitangents to a quartic curve.

If this group is contained by a larger primitive group there must be homologies not commutative with the above group and with centers in $x_1 = x_2 = x_3 = x_4 = 0$. The group on this system of points must then be of order $2 \cdot 576$ and the homologies must generate its invariant subgroup of order $2 \cdot 96$. In the S_3 containing the centers of these homologies their centers and axial planes must form the vertices and opposite faces of three tetrahedra. One vertex of each of these three tetrahedra must be left invariant by both U and V. We may choose one of these as $\xi_6 - \xi_7 = 0$. The other three vertices of this tetrahedron must then be $\xi_6 + \xi_7 = 0$, $\xi_5 + \xi_8 = 0$, $\xi_5 - \xi_8 = 0$. The invariant linear system of the homology with center $\xi_6 - \xi_7 = 0$ may be taken as $x_6 - x_7 = 0$. This homology is $x'_6 = x_7$, $x'_7 = x_6$, $x'_j = x_j$ $(j \neq 6, 7)$.

A group is then generated which permutes 120 centers of homologies. Of the 120 twelve are in $x_5 = x_6 = \ldots . = 0$ and twelve in $x_1 = x_2 = x_3 = x_4 = 0$, and the other 96 lie by pairs on the 48 lines joining the vertices of the three tetrahedra in $x_5 = x_6 = \ldots . = 0$ with the vertices of three corresponding tetrahedra in $x_1 = x_2 = x_3 = x_4 = 0$. The order of the group commutative with one of the homologies is $2^{11}\cdot 3^4\cdot 5\cdot 7$. It contains, however, the C_2 $x'_i=-x_i$, $x'_j=x_j$ $(i\leqq 8,\ j>8)$, which is invariant under the whole group. Hence the order of the whole group as a collineation group in the S_7 containing the centers of the homologies is $2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 120=2^{13}\cdot 3^5\cdot 5^2\cdot 7$.

This is the other group found by Burnside (*loc. cit.*). To put it in the form in which it is exhibited by him, we first transform it by $x'_i=-x_i$, $x'_j=x_j$ $(i\leqq 3,\ j>3)$, and then by $x'_{i+1}-x'_i=x_{i+1}-x_i$, $x'_1+x'_2+\ldots.+x'_8=-3\,(x_1+x_2+\ldots.+x_8)$, $x'_j=x_j$ $(i<8,\ j>8)$. The group then contains and may be generated by the symmetric group in the first eight variables and the homology with center $\xi_1+\xi_2+\xi_3=0$ and invariant linear system $2\,(x_1+x_2+x_3)-(x_4+x_5+x_6+x_7+x_8)=0$. The latter is obtained by the two transformations from that with center $\xi_1+\xi_2+\ldots.+\xi_8$ and invariant linear system $x_1+x_2+\ldots.+x_8=0$. It may be written:

$$x'_1=\frac{1}{3}s_1-(x_2+x_3),\ x'_2=\frac{1}{3}s_1-(x_1+x_3),\ x'_3=\frac{1}{3}s_1-(x_1+x_2),\ x'_i=x_i\ (i>3),$$

where $s_1=x_1+x_2+\ldots.+x_8$.

If when the group is exhibited in this form the coefficients of the transformations and the coordinates of the points be reduced modulo 2, the group remains the same as a permutation group on the 120 points, but it has then the quadratic invariant $\Sigma x_ix_j\equiv 0$ $(i,j=1,2,\ldots.,8;\ i\neq j)$. The group is therefore simply isomorphic with the first hypo-abelian group on eight variables.* Burnside states this result without proof.

Returning now to the original choice of coordinates, we inquire whether any larger primitive group can contain this group. If so, there must be more homologies not commutative with it having their centers in $x_1=x_2=x_3=x_4=0$. Hence there must be a group in an S_4 containing a subgroup of order $2\cdot 576$. But there is no such group. Hence we have found all groups in which every homology is commutative with at least one homology in each dihedral G_6.

§ 3. Theorem 4. *The only primitive collineation groups in* $n\,(>4)$ *variables, which contain homologies not commutative with any one of the three in a dihedral* G_6, *are of order* $2^6\cdot 3^4\cdot 5$ *and* $2^8\cdot 3^6\cdot 5\cdot 7$ *for* $n=5,6$ *respectively.*

An homology which is not commutative with any one of the three in a dihedral G_6 must generate with it a group of order $3\cdot 18$. Such a group is generated by

$$\begin{array}{lll} x'_1=x_2, & x'_2=x_1, & x'_j=x_j\quad (j>2)\,; \\ x'_1=\omega x_2, & x'_2=\omega^2x_1, & x'_j=x_j\quad (j>2)\,; \\ x'_2=x_3, & x'_3=x_2, & x'_j=x_j\quad (j\neq 2,3), \end{array}$$

where ω is a cube root of unity.

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* Jordan, *Traité des Substitutions*, pp. 195–206; Dickson, *Linear Groups*, Chap. VIII.

Consider an homology which is not commutative with this group. It must generate with it a group permuting the vertices of a tetrahedron. The group of order $3 \cdot 18$ leaves invariant four triangles in $x_4 = x_5 = \ldots . = 0$, one vertex of each being $\xi_1 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$, $\omega \xi_1 + \xi_2 + \xi_3 = 0$, $\omega^2 \xi_1 + \xi_2 + \xi_3 = 0$. We suppose that the vertices of the first triangle are vertices of the invariant tetrahedron, and we choose the fourth to be $\xi_4 = 0$. If the center of an homology be chosen as $\xi_3 - \xi_4 = 0$, the invariant system of points may be taken to be $x_3 - x_4 = 0$. This homology is then $x'_3 = x_4$, $x'_4 = x_3$, $x'_j = x_j$ $(j \neq 3, 4)$. The group generated is of order $3^3 \cdot 4!$.

As we have seen, every homology which is not commutative with the group of order $3 \cdot 18$ must permute in common with it the vertices of a tetrahedron, and the same must be true for the conjugate groups. Hence an homology which is not commutative with the group of order $3^3 \cdot 4!$ must be such that of the first four coordinates of its center either three are 0 or else the ratios are $1, \omega, \omega^2$. If the only centers present are of the former type, it is easy to show that no primitive group can exist. Since the group of order $3^3 \cdot 4!$ is invariant under any transformation which multiplies each of the first four variables by $1, \omega, \omega^2$ and makes an arbitrary substitution on the others, we may take a center of the latter type to be $\xi_1 + \xi_2 + \ldots . + \xi_6 = 0$. The invariant linear system of this homology may be taken as $x_1 + x_2 + \ldots . + x_6 = 0$. The homology may be written:

$$x'_1 + x'_2 + \ldots . + x'_6 = -(x_1 + x_2 + \ldots . + x_6),$$
$$x'_{i+1} - x'_i = x_{i+1} - x_i, \quad x'_j = x_j \quad (i < 6, \ j > 6).$$

We find that a group is generated which permutes 45 centers of homologies. Of these 45, nine are in $x_4 = x_5 = \ldots . = 0$, and the other 36 lie by threes on the 12 lines joining the vertices of the four triangles in $x_4 = x_5 = \ldots . = 0$ with four corresponding points in $x_1 = x_2 = x_3 = 0$, *i. e.*, $\xi_4 = 0$, $\xi_4 + \xi_5 + \xi_6 = 0$, $\omega \xi_4 + \xi_5 + \xi_6 = 0$, $\omega^2 \xi_4 + \xi_5 + \xi_6 = 0$. To determine the order of the group as a collineation group in the S_4 which contains the centers of the homologies, we find that there are 40 planes, each of which contains the centers of nine homologies, and hence 40 cyclic groups conjugate with that generated by $T : x'_i = \omega x_i$, $x'_j = x_j$ $(i \leqq 3, \ j > 3)$. The order of the subgroup (in the S_4) which is commutative with T is $3 \cdot 216$. It permutes the four triangles in $x_4 = x_5 = \ldots . = 0$, and on the points in $x_1 = x_2 = x_3 = 0$ it is of order 12. Each permutation of the four triangles corresponds to a particular transformation on these points. To one which leaves fixed the triangle $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$ and is of period 3 on the others, there corresponds the transformation $U : x'_4 = \omega x_4$, $x'_j = x_j$ $(j > 4)$, and to one which leaves fixed the triangle

$\xi_1+\xi_2+\xi_3=0,\ \xi_1+\omega\xi_2+\omega^2\xi_3=0,\ \xi_1+\omega^2\xi_2+\omega\xi_3=0$ and is of period 3 on the others, there corresponds the transformation

$$V: x_4'+x_5'+x_6'=\omega\,(x_4+x_5+x_6),$$
$$x_{i+1}'-x_i'=x_{i+1}-x_i,\ x_j'=x_j\ \ (i=4,5;\ j>6).$$

The order of the group as a collineation group in the S_4 is then $3\cdot 216\cdot 40 = 2^6\cdot 3^4\cdot 5$. The order of the group in any larger space containing this S_4 is, however, of order $2^7\cdot 3^4\cdot 5$, since it contains the invariant C_2: $x_i'=-x_i$, $x_5'=-x_6$, $x_6'=-x_5$, $x_j'=x_j$ $(i<5,\ j>6)$. This C_2 is the product of any five mutually commutative homologies. The group in the S_4 is readily shown to contain primitive subgroups of order $6!$.

If this group is contained by a larger primitive group, there must be homologies not commutative with it. Of the first four coordinates of the center of such an homology either three must be 0 or else the ratios must be $1, \omega, \omega^2$. Under the group commutative with T a center of the latter sort will be conjugate with centers of the former sort. Since the symmetric group on the first four variables is present, there must be centers whose first three coordinates are 0. Such homologies must be commutative with all those with centers in $x_4=x_5=\ldots\ldots=0$.

There can be no centers of homologies on the line containing the centers of the four C_3 generated by U, V, UVU^2, U^2VU. For under any group generated by homologies with centers on that line the four centers would be conjugate. Hence there would be additional centers in an S_3 such as $x_5=x_6=\ldots\ldots=0$, which is impossible.

An homology with center in $x_1=x_2=x_3=0$, but not on this line and not commutative with the above group, must generate with the four C_3 a group of order $3\cdot 216$. An homology which is commutative with V we may take to be $x_4'=x_5,\ x_5'=x_4,\ x_j'=x_j$ $(j\neq 4,5)$.

We find that a group is generated permuting 126 centers of homologies. Of the 126 centers nine are in $x_4=x_5=\ldots\ldots=0$ and nine in $x_1=x_2=x_3=0$, and the other 108 lie by threes on the 36 lines joining the vertices of the four triangles in $x_4=x_5=\ldots\ldots=0$ with the vertices of four corresponding triangles in $x_1=x_2=x_3=0$. The order of the group as a collineation group in the S_5 containing the centers of homologies is $2^7\cdot 3^4\cdot 5\cdot 126=2^8\cdot 3^6\cdot 5\cdot 7$.

This group must evidently contain a self-conjugate subgroup of index 2, consisting of all the transformations which may be written with determinant unity without introducing any irrationality other than ω. It will be shown later that the latter group is (1, 1) isomorphic with a known simple group. In any larger space containing the S_5 each of these groups contains an invariant C_6 generated by $x_i'=-\omega x_i,\ x_j'=x_j$ $(i\leqq 6,\ j>6)$.

If there is a larger primitive group containing the group of order $2^8 \cdot 3^6 \cdot 5 \cdot 7$, there must be homologies not commutative with it. The centers of some of these must lie in $x_1 = x_2 = x_3 = 0$. But the group on these points is of order $6 \cdot 216$ and no larger group can contain it. Hence we have found all collineation groups in more than four variables which contain homologies not commutative with any one of the three in a dihedral G_6.

THEOREM 5. *The primitive group of order* $2^8 \cdot 3^6 \cdot 5 \cdot 7$ *in six variables is* (1, 2) *isomorphic with the first orthogonal group on six indices with modulus* 3.*

To prove this theorem we first transform the group as written above by $x_i' = \omega x_i$, $x_j' = x_j$ $(i = 1, 2;\ j > 2)$, and then by

$$x_1' + x_2' + \ldots. + x_6' = \omega^2 (x_1 + x_2 + \ldots. + x_6),$$
$$x_{i+1}' - x_i' = x_{i+1} - x_i,\ x_j' = x_j \quad (i < 6,\ j > 6).$$

The group as transformed contains the symmetric group in the six variables, the homology $x_1' = -x_2$, $x_2' = -x_1$, $x_j' = x_j$ $(j > 2)$ and the homology with center $-\sqrt{-3}\,\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0$ and invariant linear system $\sqrt{-3}\,x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$. These two homologies are obtained from those with centers $\xi_1 + \xi_2 + \ldots. + \xi_6 = 0$ and $\omega^2 \xi_1 + \omega \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0$ respectively. The symmetric group on the six variables together with the first of these homologies generates a group of order $2^4 \cdot 6!$ under which $\xi_1 + \xi_2 = 0$ goes into 30 positions and $-\sqrt{-3}\,\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0$ into 96 positions. These are the 126 points permuted by the group.

If now the coefficients of the transformations and the coordinates of the points be reduced modulo 3, the group remains the same as a permutation group on these 126 points, but it then has the quadratic invariant $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \equiv 0$. The self-conjugate subgroup of index 2 is therefore (1, 1) isomorphic with the simple group of the same order.

The whole group has primitive subgroups in the six variables. The symmetric group on the first five variables together with the two homologies with centers $\xi_1 + \xi_2 = 0$ and $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 - \sqrt{-3}\,\xi_6 = 0$ generates a group of order $2^7 \cdot 3^4 \cdot 5$, having the invariant quadratic spread $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0$. This is the well-known group of this order and may, by a slight change of variables, be exhibited with rational coefficients. The symmetric group on the six variables together with the homology with center $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 - \sqrt{-3}\,\xi_6 = 0$ generates a group of order $7!$.

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* Jordan, *Traité des Substitutions*, pp. 161–170; Dickson, *Linear Groups*, Chap. VII.